

Asymptotic behavior and critical coupling in the scalar Yukawa model from Schwinger-Dyson equations

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Abstract. A sequence of n -particle approximations for the system of Schwinger-Dyson equations is investigated in the model of a complex scalar field ϕ and a real scalar field χ with the interaction $g\phi^*\phi\chi$. In the first non-trivial two-particle approximation, the system is reduced to a system of two nonlinear integral equations for propagators. The study of this system shows that for equal masses a critical coupling constant g_c^2 exists, which separates the weak- and strong-coupling regions with the different asymptotic behavior for deep Euclidean momenta. In the weak-coupling region ($g^2 < g_c^2$), the propagators are asymptotically free, which corresponds to the wide-spread opinion about the dominance of perturbation theory for this model. At the critical point the asymptotics of propagators are $\sim 1/p$. In the strong coupling region ($g^2 > g_c^2$), the propagators are asymptotically constant, which corresponds to the ultra-local limit. For unequal masses, the critical point transforms into a segment of values, in which there are no solutions with a self-consistent ultraviolet behavior without Landau singularities.

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1 Introduction

A system of Schwinger-Dyson equations (SDEs) is the infinite set of integral equations containing, in principle, all the information about a model of quantum field theory. So far, there are no effective methods for the study of this infinite system as a whole, so it is necessary to truncate this system with a finite set of equations. This truncation usually refers to either an expansion in a small parameter (examples of such expansions are the coupling-constant perturbation theory and the $1/N$ -expansion), or a simulation (and rather, guessing) of some properties of the model (see, e.g., [1]). An initial truncation of the system determines further approximations of the complete system, which is thus attached to the leading approximation.

The system of SDEs is, in fact, the system of relations between derivatives of the generating functional of Green functions, resulting from the functional-differential SDE, which acts as a dynamical principle of the theory. If we approximate the generating functional with the first n terms of an expansion in powers of a source, the system of SDEs can be approximated by a closed system of integral equations. This system defines the n -th term of a sequence of approximations, which for $n \rightarrow \infty$ obviously goes into the complete system of SDEs. A sequence of such approximations in the model of complex scalar field with interaction $\lambda(\phi^*\phi)^2$ was considered in [2]. For the simplest nontrivial approximation (the “two-particle approximation”) an asymptotic solution of the corresponding system in deep Euclidean region was obtained. In the strong-coupling region, this solution is free from Landau singularities. In [3], in the same approximation, the Yukawa model was considered, for which a self-consistent solution without Landau singularities in the Euclidean region also has been constructed.

In this paper, in the same approximation, we consider a system of SDEs in the model of the complex scalar field ϕ (phion) and real scalar field χ (chion) with the interaction $g\phi^*\phi\chi$ in four dimensions. This model, also known as a scalar Yukawa model, is used in nuclear physics as a simplified version of the Yukawa model without spin degrees of freedom, as well as an effective model of the interaction of scalar quarks [4]. Despite its well-known imperfection associated with its instability [5] (or more precisely, the metastability [6] – [9]), this model, as the simplest model of the interaction of fields, often used as a prototype of more substantive theories to elaborate the various non-perturbative approaches in the quantum field theory.

In the two-particle approximation, the system of SDEs for the scalar Yukawa model is a system of two nonlinear integral equations for propagators. The study of this system shows a change of the asymptotic behavior of propagators in the deep Euclidean region in a vicinity of a certain critical value of the coupling constant. For small values of the coupling the propagators behave as free, which is consistent with the wide-spread opinion about the dominance of perturbation theory for this super-renormalizable model. In the strong-coupling region, however, the asymptotic behavior changes dramatically – both propagators in the deep Euclidean region tend to some constant limits.

The existence of a critical coupling constant in the scalar Yukawa model was noticed by practically all authors who have investigated this model using different methods (see, e.g., [10]-[16] and references therein). This critical constant is generally regarded as a limit on the coupling constant for a self-consistent description of the model by some method. In our approach, however, the self-consistent solution for propagators exists also for the strong coupling, and the existence of the critical coupling looks more like as a phase transition in accordance with the general definition of the phase transition as a sharp change of properties of the model with a smooth change of parameters (see, e.g., [17]).

The structure of the paper is as follows. In section 2, the general formalism of SDEs for the scalar Yukawa model is described and the necessary definitions and notations are given. A simplest non-perturbative expansion, namely the mean-field expansion is also considered in this section. The leading term of this expansion corresponds to the chain summation for a chion propagator. The problem of restoring of crossing properties for this expansion is discussed, and the existence of a critical value g_c^2 of the coupling is shown. For $g^2 > g_c^2$, the asymptotics of inverse chion propagator in deep Euclidean region becomes negative, which leads to the appearance of a Landau singularity and the associated violation of the self-consistency of the method.

In section 3, a construction of n -particle approximations for the system of SDEs is given, and the two-particle approximation is considered in detail. The system of SDEs in the two-particle approximation is reduced to a system of nonlinear integral equations for propagators. The study of the system at large Euclidean momenta shows that for the equal masses of fields a self-consistent solution exists for each value of the coupling, and there is a critical coupling value, which separates the weak and strong coupling regions with a different asymptotic behavior. For the unequal masses, the weak and strong coupling regions are separated by the segment of the intermediate coupling without self-consistent solutions.

In section 4, a three-particle approximation for the model is briefly considered, and the crossing-symmetry problem for two-particle amplitude is discussed. A discussion of the results is contained in section 5. In the Appendix, the exact solution of linearized integral equation in the strong-coupling region is obtained for the case of equal masses.

2 SDEs and the mean-field expansion in the bilocal source formalism

2.1 Preliminaries

We consider the model of interaction of a complex scalar field ϕ (phion) and a real scalar field χ (chion) with the Lagrangian

$$\mathcal{L} = -\partial_\mu \phi^* \partial_\mu \phi - m_0^2 \phi^* \phi - \frac{1}{2}(\partial_\mu \chi)^2 - \frac{\mu^2}{2}\chi^2 + g\phi^* \phi \chi \quad (1)$$

in a four-dimensional Euclidean space ($x \in E_4$). The coupling constant g has a dimension of mass.

The generating functional of Green functions (vacuum averages) is the functional integral

$$G(\eta, j) = \int D(\phi, \phi^*, \chi) \exp \left\{ \int dx \mathcal{L}(x) - \int dx dy \phi^*(y) \eta(y, x) \phi(x) + \int dx j(x) \chi(x) \right\}. \quad (2)$$

Here, η is a bilocal source of the phion field,² j is a single source of the chion field.

The translational invariance of the functional integration measure leads to the functional-differential SDEs for the generating functional G . In terms of the logarithm $Z = \log G$ these equations are

$$g \left[\frac{\delta^2 Z}{\delta \eta(y, x) \delta j(x)} + \frac{\delta Z}{\delta j(x)} \frac{\delta Z}{\delta \eta(y, x)} \right] = (m_0^2 - \partial_x^2) \frac{\delta Z}{\delta \eta(y, x)} + \int dx_1 \eta(x, x_1) \frac{\delta Z}{\delta \eta(y, x_1)} + \delta(x - y), \quad (3)$$

$$\frac{\delta Z}{\delta j(x)} = \int dx_1 D_c(x - x_1) j(x_1) - g \int dx_1 D_c(x - x_1) \frac{\delta Z}{\delta \eta(x_1, x_1)}. \quad (4)$$

Here $D_c \equiv (\mu^2 - \partial^2)^{-1}$.

Equation (4) allows us to express all Green functions with chion legs in terms of functions that contain phions only. Thus, the differentiation of (4) over η gives us the three-point function

$$V(x, y|z) \equiv - \frac{\delta^2 Z}{\delta j(z) \delta \eta(y, x)} \Big|_{\eta=j=0} = g \int dz_1 D_c(z - z_1) Z_2 \left(\begin{array}{cc} z_1 & z_1 \\ x & y \end{array} \right), \quad (5)$$

where

$$Z_2 \left(\begin{array}{cc} x & y \\ x' & y' \end{array} \right) \equiv \frac{\delta^2 Z}{\delta \eta(y', x') \delta \eta(y, x)} \Big|_{\eta=j=0} \quad (6)$$

is the two-particle phion function. The differentiation of (4) over j with taking into account equation (5) gives us the chion propagator:

$$D(x - y) \equiv \frac{\delta^2 Z}{\delta j(y) \delta j(x)} \Big|_{\eta=j=0} = D_c(x - y) + g^2 \int dx_1 dy_1 D_c(x - x_1) Z_2 \left(\begin{array}{cc} x_1 & x_1 \\ y_1 & y_1 \end{array} \right) D_c(y_1 - y), \quad (7)$$

etc.

Excluding with the help of the SDE (4), a differentiation over j in SDE (3), we obtain at $j = 0$ the SDE for the generating functional:

$$g^2 \int dx_1 D_c(x - x_1) \left[\frac{\delta^2 Z}{\delta \eta(x_1, x_1) \delta \eta(y, x)} + \frac{\delta Z}{\delta \eta(x_1, x_1)} \frac{\delta Z}{\delta \eta(y, x)} \right] + (m_0^2 - \partial_x^2) \frac{\delta Z}{\delta \eta(y, x)} + \int dy_1 \eta(x, y_1) \frac{\delta Z}{\delta \eta(y, y_1)} + \delta(x - y) = 0, \quad (8)$$

²The formalism of the bilocal source was first elaborated in the quantum field theory by Dahmen and Jona-Lasinio [18]. We consider this using as a convenient choice of the functional variable.

which only contains the derivatives over the bilocal source η .

Sequential differentiations of this equation give us the infinite system of SDEs for Green functions. For our purposes, it will need the first three equations of this system. Switching off the source in equation (8), we obtain the equation

$$(m^2 - \partial_x^2)\Delta(x-y) = \delta(x-y) + g^2 \int dx_1 D_c(x-x_1) Z_2 \begin{pmatrix} x & y \\ x_1 & x_1 \end{pmatrix}. \quad (9)$$

Here $m^2 \equiv m_0^2 - \frac{g^2}{\mu^2}\Delta(x=0)$ and

$$\Delta(x-y) \equiv -\frac{\delta Z}{\delta\eta(y,x)} \Big|_{\eta=0} \quad (10)$$

is the pion propagator.

A differentiation over η gives us the second equation:

$$(m^2 - \partial_x^2)Z_2 \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} - g^2 \int dx_1 D_c(x-x_1) Z_2 \begin{pmatrix} x_1 & x_1 \\ x' & y' \end{pmatrix} \Delta(x-y) + \\ + g^2 \int dx_1 D_c(x-x_1) Z_3 \begin{pmatrix} x_1 & x_1 \\ x & y \\ x' & y' \end{pmatrix} = \delta(x-y')\Delta(x'-y). \quad (11)$$

The repeated differentiation over η gives one more equation:

$$(m^2 - \partial_x^2)Z_3 \begin{pmatrix} x & y \\ x' & y' \\ x'' & y'' \end{pmatrix} - g^2 \int dx_1 D_c(x-x_1) Z_3 \begin{pmatrix} x_1 & x_1 \\ x' & y' \\ x'' & y'' \end{pmatrix} \Delta(x-y) + \\ + g^2 \int dx_1 D_c(x-x_1) Z_4 \begin{pmatrix} x_1 & x_1 \\ x & y \\ x' & y' \\ x'' & y'' \end{pmatrix} = -\left\{ \delta(x-y'')Z_2 \begin{pmatrix} x' & y' \\ x'' & y \end{pmatrix} + \right. \\ \left. + g^2 \int dx_1 D_c(x-x_1) Z_2 \begin{pmatrix} x_1 & x_1 \\ x' & y' \end{pmatrix} Z_2 \begin{pmatrix} x'' & y'' \\ x & y \end{pmatrix} \right\} - \left\{ x' \leftrightarrow x'', y' \leftrightarrow y'' \right\}. \quad (12)$$

Here $Z_n \equiv \frac{\delta^n Z}{\delta\eta^n} \Big|_{\eta=0}$ is the n -particle pion function.

2.2 Kernel and inverse kernel

Integral equations with the kernel

$$K_{ab} \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} = \delta(x-x')\delta(y-y') - g^2 \delta(x'-y') \int dx_1 \Delta_a(x-x_1) D_c(x_1-y') \Delta_b(x_1-y) \quad (13)$$

will be repeatedly considered below . Here Δ_a and Δ_b are the given functions. It is easy to verify that the inverse kernel has the form

$$K_{ab}^{-1} \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} = \delta(x-x')\delta(y-y') + \delta(x'-y') \int dx_1 \Delta_a(x-x_1) f(x_1-y') \Delta_b(x_1-y), \quad (14)$$

i.e.

$$\int dx_1 dy_1 K_{ab}^{-1} \begin{pmatrix} x & y \\ x_1 & y_1 \end{pmatrix} K_{ab} \begin{pmatrix} x_1 & y_1 \\ x' & y' \end{pmatrix} = \delta(x-x')\delta(y-y') \quad (15)$$

(no summation over a and b !).

In equation (14),

$$f^{-1}(x-y) = \frac{1}{g^2} D_c^{-1}(x-y) - L_{ab}(x-y), \quad (16)$$

where

$$L_{ab}(x-y) = \Delta_a(x-y)\Delta_b(y-x). \quad (17)$$

Note also the useful property of the inverse kernel:

$$g^2 \int dx_1 D_c(x-x_1) K_{ab}^{-1} \begin{pmatrix} x_1 & x_1 \\ x' & y' \end{pmatrix} = f(x-y')\delta(x'-y'). \quad (18)$$

2.3 Mean-field expansion

The mean-field expansion for the generating functional (see [2], [3] and references therein)

$$Z = Z^{(0)} + Z^{(1)} + \dots \quad (19)$$

is based on the leading approximation:

$$g^2 \int dx_1 D_c(x-x_1) \frac{\delta Z^{(0)}}{\delta \eta(x_1, x_1)} \frac{\delta Z^{(0)}}{\delta \eta(y, x)} + (m^2 - \partial_x^2) \frac{\delta Z^{(0)}}{\delta \eta(y, x)} + \int dy_1 \eta(x, y_1) \frac{\delta Z^{(0)}}{\delta \eta(y, y_1)} + \delta(x-y) = 0. \quad (20)$$

The NLO equation is

$$g^2 \int dx_1 D_c(x-x_1) \frac{\delta Z^{(1)}}{\delta \eta(x_1, x_1)} \frac{\delta Z^{(0)}}{\delta \eta(y, x)} + g^2 \int dx_1 D_c(x-x_1) \frac{\delta Z^{(0)}}{\delta \eta(x_1, x_1)} \frac{\delta Z^{(1)}}{\delta \eta(y, x)} + (m^2 - \partial_x^2) \frac{\delta Z^{(1)}}{\delta \eta(y, x)} + \int dy_1 \eta(x, y_1) \frac{\delta Z^{(1)}}{\delta \eta(y, y_1)} = -g^2 \int dx_1 D_c(x-x_1) \frac{\delta^2 Z^{(0)}}{\delta \eta(x_1, x_1) \delta \eta(y, x)}, \quad (21)$$

etc.

Equation (20), after switching off the source, gives us the leading-order pion propagator. In the momentum space

$$\Delta_0^{-1}(p) = m^2 + p^2. \quad (22)$$

A differentiation of the leading-order equation over η gives us the equation for the leading-order two-particle function $Z_2^{(0)}$, which can be written as:

$$\int dx_1 dy_1 K_{00} \begin{pmatrix} x & y \\ x_1 & y_1 \end{pmatrix} Z_2^{(0)} \begin{pmatrix} x_1 & y_1 \\ x' & y' \end{pmatrix} = \Delta_0(x - y') \Delta_0(x' - y), \quad (23)$$

where K_{00} is kernel (13) at $\Delta_a = \Delta_b = \Delta_0$. In correspondence with equation (14), the solution of this equation is

$$Z_2^{(0)} \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} = \Delta_0(x - y') \Delta_0(x' - y) + \int dx_1 dx_2 \Delta_0(x - x_1) \Delta_0(x' - x_2) f_0(x_1 - x_2) \Delta_0(x_1 - y) \Delta_0(x_2 - y'), \quad (24)$$

where $f_0^{-1} = g^{-2} D_c^{-1} - L_0$, and $L_0 = \Delta_0 \Delta_0$ is the scalar loop. Equation (7) after simple calculations with taking into account above formulae gives us the leading-order chion propagator D_0 . In the momentum space

$$D_0^{-1}(p) = \mu^2 + p^2 - g^2 L_0(p^2). \quad (25)$$

The repeated differentiation of the leading-order equation over η gives us the equation for the leading-order three-particle function whose solution is

$$Z_3^{(0)} \begin{pmatrix} x & y \\ x' & y' \\ x'' & y'' \end{pmatrix} = \int dx_1 dy_1 K_{00}^{-1} \begin{pmatrix} x & y \\ x_1 & y_1 \end{pmatrix} Z_{30} \begin{pmatrix} x_1 & y_1 \\ x' & y' \\ x'' & y'' \end{pmatrix}. \quad (26)$$

Here

$$Z_{30} \begin{pmatrix} x & y \\ x' & y' \\ x'' & y'' \end{pmatrix} \equiv - \left\{ \delta(x - y') Z_2^{(0)} \begin{pmatrix} x' & y \\ x'' & y'' \end{pmatrix} + g^2 \int dx_1 D_c(x - x_1) Z_2^{(0)} \begin{pmatrix} x_1 & x_1 \\ x' & y' \end{pmatrix} Z_2^{(0)} \begin{pmatrix} x & y \\ x'' & y'' \end{pmatrix} \right\} - \left\{ x' \leftrightarrow x'', y' \leftrightarrow y'' \right\}, \quad (27)$$

Note the following property of the three-particle function resulting from equation (18):

$$g^2 \int dx_1 D_c(x - x_1) Z_3^{(0)} \begin{pmatrix} x_1 & x_1 \\ x & y \\ x' & y' \end{pmatrix} = \int dx_1 f_0(x - x_1) Z_{30} \begin{pmatrix} x_1 & x_1 \\ x & y \\ x' & y' \end{pmatrix}. \quad (28)$$

The other functions of the leading approximation can be calculated in the same manner.

Note that in contrast to the functional derivatives of Z over single source j , the higher derivatives of Z over bilocal source η are not connected parts of corresponding many-particle functions. Thus, the two-particle phion function Z_2 is not the connected part Z_2^c of two-particle and related to it by the formula:

$$Z_2 \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} = \Delta(x - y')\Delta(x' - y) + Z_2^c \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} \quad (29)$$

A characteristic feature of many-particle functions of the leading approximation is their incomplete structure in terms of crossing symmetry. The Bose symmetry of the theory dictates the crossing symmetry of the connected part Z_2^c :

$$Z_2^c \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} = Z_2^c \begin{pmatrix} x' & y' \\ x & y \end{pmatrix} = Z_2^c \begin{pmatrix} x' & y \\ x & y' \end{pmatrix} = Z_2^c \begin{pmatrix} x & y' \\ x' & y \end{pmatrix}. \quad (30)$$

It is easy to see that $Z_2^{(0)c}$ (this is the second term on the rhs of equation (24)) satisfies the first equality in (30), but breaks the other two. This apparent discrepancy is a feature of many non-perturbative approximations. It is inherent, for example, to the Bethe-Salpeter equation in the ladder approximation. We face similar problems in the two-particle approximation, which will be considered below. To restore the missing crossing symmetry of the leading approximation, it is necessary to look at the next order.

Calculations in the next order quite similar to the above. Equation (21) with the source being switched off gives us the equation for the first correction Δ_1 to the propagator. The differentiation of equation (21) over the source gives the equation for the NLO two-particle function $Z_2^{(1)}$, which can be written as an equation with kernel K_{00} , and taking into account the above formulae the solution is

$$Z_2^{(1)} \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} = Z_{21} \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} + \int dx_1 dx_2 \Delta_0(x - x_1) f_0(x_1 - x_2) \Delta_0(x_1 - y) Z_{21} \begin{pmatrix} x_2 & x_2 \\ x' & y' \end{pmatrix}, \quad (31)$$

where

$$\begin{aligned} Z_{21} \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} &\equiv -g^2 \int dx_1 dy_1 \Delta_0(x - y_1) D_c(y_1 - x_1) Z_3^{(0)} \begin{pmatrix} x_1 & x_1 \\ y_1 & y \\ x' & y' \end{pmatrix} + \\ &+ \Delta_0(x - y') \Delta_1(x' - y) - \frac{g^2}{\mu^2} \Delta_1(x = 0) \int dx_1 \Delta_0(x - x_1) Z_2^{(0)} \begin{pmatrix} x_1 & y \\ x' & y' \end{pmatrix} + \\ &+ g^2 \int dx_1 dx_2 \Delta_0(x - x_1) D_c(x_1 - x_2) \Delta_1(x_1 - y) Z_2^{(0)} \begin{pmatrix} x_2 & x_2 \\ x' & y' \end{pmatrix}. \end{aligned} \quad (32)$$

From these equations, together with equation (28), it follows that $Z_2^{(1)}$ contains the term

$$\int dx_1 dx_2 \Delta_0(x - x_1) \Delta_0(x' - x_2) f_0(x_1 - x_2) \Delta_0(x_2 - y) \Delta_0(x_1 - y') =$$

$$= Z_2^{(0)c} \begin{pmatrix} x & y' \\ x' & y \end{pmatrix} = Z_2^{(0)c} \begin{pmatrix} x' & y \\ x & y' \end{pmatrix}, \quad (33)$$

which restores the missing crossing symmetry of the LO two-particle function. Such restoration of the crossing symmetry is typical for non-perturbative expansions in the formalism of bilocal source (for similar examples in other models see [19], [20]).

The above equations contain divergent integrals and require a renormalization. According to the standard recipe we introduce the renormalized propagators Δ_r and D_r and impose on them the normalization condition³

$$\Delta_r^{-1}(0) = m_r^2, \quad \left. \frac{d\Delta_r^{-1}}{dp^2} \right|_{p=0} = 1 \quad (34)$$

and

$$D_r^{-1}(0) = \mu_r^2, \quad \left. \frac{dD_r^{-1}}{dp^2} \right|_{p=0} = 1. \quad (35)$$

In the leading approximation of the mean-field expansion the pion propagator is given by equation (22), which implies that it is sufficient to replace $\Delta_0 \rightarrow \Delta_r$, $m_r^2 \rightarrow m^2$, i.e., the renormalized leading-order pion propagator is

$$\Delta_r^{-1}(p) = m_r^2 + p^2. \quad (36)$$

For the renormalized chion propagator according with equation (25), we obtain in the leading approximation:

$$D_r^{-1}(p^2) = \mu_r^2 + p^2 - g^2 L_r(p^2), \quad (37)$$

where $L_r(p^2) = L_0(p^2) - L_0(0) - p^2 L'_0(0)$.

At $p^2 \rightarrow \infty$,

$$D_r^{-1}(p^2) = \left(1 - \frac{g_r^2}{96\pi^2 m_r^2}\right) p^2 + O(\log \frac{p^2}{m_r^2}). \quad (38)$$

As can be seen from this equation, the asymptotic behavior of the pion propagator in the deep Euclidean region is self-consistent in the region of weak coupling which in this case is determined by the condition $g^2 < g_c^2 = 96\pi^2 m_r^2$. When $g^2 > g_c^2$, the asymptotics of the inverse propagator becomes negative, which leads to the appearance of a Landau singularity for the chion propagator in the Euclidean region, which in turn means a violation of basic physical principles. As was mentioned in the Introduction, the existence of similar restrictions on the value of the coupling constant in the model has been noted by many authors. Some authors (see, e.g.,

³For the easement of the following calculations we choose the normalization point at zero momenta. Such a choice is not a big deal for simple calculations of the mean-field expansion, but it is highly significant for more complicated calculations in a more pithy two-particle approximation (see below).

[14], [15]) believe self-evident that the presence of this kind of critical constants reflects the metastability of the model. It seems to us, however, that the presence of such a singularity means firstly the non-applicability of the calculation method in the strong coupling region and the need for more meaningful non-perturbative approximations.

3 Two-particle approximation

The system of SDEs, generated by functional-differential equation (8), is an infinite set of integral equations for n -particle phion functions $Z_n \equiv \delta^n Z / \delta \eta^n|_{\eta=0}$. The first three SDEs are equations (9), (11) and (12). The n th SDE is the $(n-1)$ th derivative of SDE (8) with the source being switched off and includes a set of functions from one-particle function (phion propagator) to the $(n+1)$ -particle phion function. In order to obtain a sequence of closed systems of equations, we proceed as follows. We call "the n -particle approximation of the system of SDEs" the system of n SDEs, in which the first $n-1$ equations are exact and the n th SDE is truncated by omitting the $(n+1)$ -particle function. It is evident that the sequence of such approximations goes to the exact set of SDEs at $n \rightarrow \infty$. The one-particle approximation is simply equation (9) without Z_2 . This approximation has a trivial solution which is a free propagator. The two-particle approximation is a system of equation (9) and equation (11) without Z_3 :

$$(m^2 - \partial_x^2) Z_2 \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} - g^2 \int dx_1 D_c(x-x_1) Z_2 \begin{pmatrix} x_1 & x_1 \\ x' & y' \end{pmatrix} \Delta(x-y) = \delta(x-y') \Delta(x'-y). \quad (39)$$

This system of equations will be the object of the present investigation.

An alternative method to obtain this system is a modification of the mean-field expansion of section 2. In this modified mean-field expansion, equations (9) and (39) are the basic equations of the expansion. A construction of such modified mean-field expansion was performed for the model $\lambda(\phi^* \phi)^2$ in work [2] and easily can be extended to the considered model.

In framework of the two-particle approximation. Z_2 can be expressed as a functional of Δ . In fact, equation (39) can be written as

$$\int dx_1 dy_1 K \begin{pmatrix} x & y \\ x_1 & y_1 \end{pmatrix} Z_2 \begin{pmatrix} x_1 & y_1 \\ x' & y' \end{pmatrix} = \Delta_c(x-y') \Delta(x'-y), \quad (40)$$

where $K = K_{ab}$ at $\Delta_a = \Delta_c \equiv (m^2 - \partial^2)^{-1}$ and $\Delta_b = \Delta$ (see (13)). In correspondence with equation (14), we obtain

$$Z_2 \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} = \Delta_c(x-y') \Delta(x'-y) + \int dx_1 dx_2 \Delta_c(x-x_1) \Delta(x'-x_2) f(x_1-x_2) \Delta(x_1-y) \Delta_c(x_2-y'), \quad (41)$$

where f is given by equation (16) and

$$L(x-y) = \Delta_c(x-y)\Delta(y-x). \quad (42)$$

Equations (7) and (9) with the taking into account above formulae and equation (18) give us the system of equations for the chion and phion propagators. In the momentum space, this system has the form

$$\begin{cases} D^{-1}(p^2) = \mu^2 + p^2 - g^2 L(p^2) \\ \Delta^{-1}(p^2) = m^2 + p^2 - g^2 K(p^2), \end{cases} \quad (43)$$

where

$$L(p^2) = \int \frac{d^4 q}{(2\pi)^4} \Delta_c(p+q)\Delta(q), \quad K(p^2) = \int \frac{d^4 q}{(2\pi)^4} \Delta_c(p-q)D(q). \quad (44)$$

The renormalization of equations of the two-particle approximation is similar to the renormalization of equations of the mean-field expansion with the imposition of normalization conditions (34) and (35) on the renormalized propagators. The system of renormalized equations for the chion and phion propagators is:

$$\begin{cases} D_r^{-1}(p^2) = \mu_r^2 + p^2 - g^2 L_r(p^2) \\ \Delta_r^{-1}(p^2) = m_r^2 + p^2 - g^2 K_r(p^2), \end{cases} \quad (45)$$

where

$$L_r = L(p^2) - L(0) - p^2 L'(0), \quad K_r = K(p^2) - K(0) - p^2 K'(0). \quad (46)$$

The system of equations (45) is a system of non-linear integral equations, which is quite difficult for the analytical study. In the study of the asymptotic behavior in the deep Euclidean region, one can make the approximation, which greatly simplifies calculations, namely to replace Δ_c by the asymptotics at $p^2 \rightarrow \infty$, i.e. by the massless propagator $1/p^2$. This massless integration approximation is quite usual in investigations in the deep Euclidean region [21].

Using the well-known formula

$$\int \frac{d^4 q}{(2\pi)^4} \frac{\Phi(q^2)}{(p-q)^2} = \frac{1}{16\pi^2} \left[\frac{1}{p^2} \int_0^{p^2} \Phi(q^2) q^2 dq^2 + \int_{p^2}^\infty \Phi(q^2) dq^2 \right] \quad (47)$$

gives us the system of equations for propagators in the massless integration approximation:

$$\begin{cases} \Delta^{-1}(p^2) = m^2 + (1 - \frac{q^2}{32\pi^2 \mu^2}) p^2 + \frac{q^2}{16\pi^2} \int_0^{p^2} dq^2 D(q^2) (1 - \frac{q^2}{p^2}) \\ D^{-1}(p^2) = \mu^2 + (1 - \frac{q^2}{32\pi^2 m^2}) p^2 + \frac{q^2}{16\pi^2} \int_0^{p^2} dq^2 \Delta(q^2) (1 - \frac{q^2}{p^2}). \end{cases} \quad (48)$$

Here and below we omit the index r , bearing in mind that all quantities are renormalized.

Note that the system of equations (48) is symmetric with respect to the change

$$\Delta \longleftrightarrow D, \quad m^2 \longleftrightarrow \mu^2. \quad (49)$$

With taking into account this symmetry, we introduce the dimensionless variables:

$$u = \frac{\Delta^{-1}}{m^2}, \quad v = \frac{D^{-1}}{\mu^2}, \quad t = \frac{p^2}{\mu m}, \quad t' = \frac{q^2}{\mu m}, \quad \lambda = \frac{g^2}{32\pi^2 \mu m}.$$

(Here $m \equiv \sqrt{m^2}$, $\mu \equiv \sqrt{\mu^2}$.)

In terms of these variables, the system of equations (48) takes the form

$$\begin{cases} u(t) = (\frac{\mu}{m} - \lambda)t + 1 + 2\lambda \int_0^t \frac{dt'}{v(t')} (1 - \frac{t'}{t}) \\ v(t) = (\frac{m}{\mu} - \lambda)t + 1 + 2\lambda \int_0^t \frac{dt'}{u(t')} (1 - \frac{t'}{t}). \end{cases} \quad (50)$$

The normalization conditions (34) and (35) for the dimensionless functions u and v have the form

$$u(0) = v(0) = 1; \quad \dot{u}(0) = \frac{\mu}{m}, \quad \dot{v}(0) = \frac{m}{\mu}. \quad (51)$$

From the system of integral equations (50) one can easily go to the system of nonlinear differential equations for the functions u and v :

$$\begin{cases} \frac{d^2}{dt^2}(tu) = 2(\frac{\mu}{m} - \lambda) + \frac{2\lambda}{v} \\ \frac{d^2}{dt^2}(tv) = 2(\frac{m}{\mu} - \lambda) + \frac{2\lambda}{u}. \end{cases} \quad (52)$$

The boundary conditions for this system are the normalization condition (51).

If v (or u) increases in the absolute value at $t \rightarrow \infty$, then for $\lambda \neq m/\mu$, $\lambda \neq \mu/m$, we obtain from the system (52) the asymptotic behavior in the deep Euclidean region:

$$\begin{cases} u = (\frac{\mu}{m} - \lambda)t + O(\log t) \\ v = (\frac{m}{\mu} - \lambda)t + O(\log t). \end{cases} \quad (53)$$

This asymptotic behavior is self-consistent in the weak-coupling region at $\lambda < \min\{\frac{\mu}{m}, \frac{m}{\mu}\}$ ($\frac{g^2}{32\pi^2} < \min\{\mu^2, m^2\}$) and corresponds to the asymptotically-free behavior of propagators:

$$\Delta \sim 1/p^2, \quad D \sim 1/p^2. \quad (54)$$

Outside the weak-coupling region one should distinguish three cases:

$\mu^2 = m^2$, $\mu^2 > m^2$ and $\mu^2 < m^2$.

At $\mu^2 = m^2$, the system of equations (50) takes the form:

$$\begin{cases} u = (1 - \lambda)t + 1 + 2\lambda \int_0^t \frac{dt'}{v(t')} (1 - \frac{t'}{t}) \\ v = (1 - \lambda)t + 1 + 2\lambda \int_0^t \frac{dt'}{u(t')} (1 - \frac{t'}{t}). \end{cases} \quad (55)$$

An iterative solution of this system gives us $u^{(n)} = v^{(n)}$ for any finite number n of iterations; hence, $u = v$ (provided the convergence of the iteration procedure). Unfortunately, it is difficult to prove the convergence of this nonlinear iteration

scheme. Nevertheless, we will assume the equality of u and v in the study of the case $\mu^2 = m^2$, leaving the possibility of an alternative for a subsequent study.

At $u = v$, the system of equations (55) is reduced to the single integral equation

$$u = (1 - \lambda)t + 1 + 2\lambda \int_0^t \frac{dt'}{u(t')} \left(1 - \frac{t'}{t}\right), \quad (56)$$

which, in turn, can be reduced to the differential equation

$$\frac{d^2}{dt^2}(tu) = 2(1 - \lambda) + \frac{2\lambda}{u}. \quad (57)$$

In the strong-coupling region $\lambda > 1$, equation (57) has the positive exact solution

$$u_{exact} = u_0 = \frac{\lambda}{\lambda - 1}. \quad (58)$$

Note that u_0 is a solution of the integral equation

$$u_0 = (1 - \lambda)t + \frac{1}{1 - \frac{1}{\lambda}} + 2\lambda \int_0^t \frac{dt'}{u_0} \left(1 - \frac{t'}{t}\right), \quad (59)$$

which differs from equation (56) by the inhomogeneous term, and this difference is small for large t in comparison with the leading part $(1 - \lambda)t$ of the inhomogeneous term. This points that u_0 is an asymptotic solution of equation (56) at $t \rightarrow \infty$. In fact, performing the linearization:

$$u = u_0 + u_1, \quad \frac{1}{u} \simeq \frac{1}{u_0} - \frac{u_1}{u_0^2}, \quad (60)$$

we obtain for u_1 the linear differential equation

$$\frac{d^2}{dt^2}(tu_1) = -a^2 u_1, \quad (61)$$

where

$$a = (\lambda - 1) \sqrt{\frac{2}{\lambda}}. \quad (62)$$

The solution of the linearized equation (61) is

$$u_1 = \frac{1}{\sqrt{t}} \left(A_1 J_1(2a\sqrt{t}) + A_2 Y_1(2a\sqrt{t}) \right). \quad (63)$$

Here J_1 Y_1 are the cylindrical functions.⁴ Therefore, $u_1 = o(u_0)$ at large t , and constant u_0 is really the asymptotic solution in the strong-coupling region.

⁴For exact solution of the linearized integral equation see Appendix.

At the critical value $\lambda = 1$, the differential equation (57) has the exact solution

$$u_{exact} = \bar{u} = \sqrt{\frac{8t}{3}}. \quad (64)$$

\bar{u} is the solution of integral equation

$$\bar{u} = 2 \int_0^t \frac{dt'}{\bar{u}} \left(1 - \frac{t'}{t}\right), \quad (65)$$

which differs from equation (56) at $\lambda = 1$ by the inhomogeneous term. It is an indication that \bar{u} is the asymptotic solution of equation (56) in the critical point $\lambda = 1$, since this inhomogeneous term is $o(\bar{u})$ at $t \rightarrow \infty$. Indeed, performing the linearization

$$u = \bar{u} + u_1, \quad \frac{1}{u} \simeq \frac{1}{\bar{u}} - \frac{u_1}{\bar{u}^2}, \quad (66)$$

we obtain the linearized equation for u_1 :

$$\frac{d^2}{dt^2}(tu_1) = -\frac{3}{4t}u_1, \quad (67)$$

which is an Euler equation, and the real solution of this equation is

$$u_1 = \frac{1}{\sqrt{t}} \left(A_1 \sin \frac{\log t}{\sqrt{2}} + A_2 \cos \frac{\log t}{\sqrt{2}} \right). \quad (68)$$

Therefore, $u_1 = o(\bar{u})$ at $t \rightarrow \infty$.

Thus, the asymptotic behavior of propagators at the critical point $\lambda = 1$ has the form

$$\Delta \sim 1/\sqrt{p^2}, \quad D \sim 1/\sqrt{p^2}. \quad (69)$$

For unequal masses, we will consider taking into account the symmetry (49) only the case $\mu^2 > m^2$. In this case, there are two critical couplings: $\lambda_{c1} = m/\mu$ and $\lambda_{c2} = \mu/m$, which determine the weak-coupling region ($\lambda < m/\mu$), the intermediate region ($m/\mu < \lambda < \mu/m$) and the strong-coupling region ($\lambda > \mu/m$).

At $\lambda = \lambda_{c1} = m/\mu$, as in the weak-coupling region, there is a self-consistent solution with the asymptotic behavior for large t of the form

$$u \simeq \frac{\mu^2 - m^2}{\mu m} t, \quad v \simeq \frac{2m^2}{\mu^2 - m^2} \log t. \quad (70)$$

In the strong-coupling region $\lambda > \lambda_{c2}$, the system of differential equations (52) has the positive exact solution

$$u_{exact} = u_0 = \frac{\lambda \mu}{\lambda \mu - m}, \quad v_{exact} = v_0 = \frac{\lambda m}{\lambda m - \mu}. \quad (71)$$

$\{u_0, v_0\}$ is the solution of the system of integral equations

$$\begin{cases} u_0 = (\frac{\mu}{m} - \lambda)t + \frac{1}{1-\frac{m}{\lambda\mu}} + 2\lambda \int_0^t \frac{dt'}{v_0} (1 - \frac{t'}{t}) \\ v_0 = (\frac{m}{\mu} - \lambda)t + \frac{1}{1-\frac{\mu}{\lambda m}} + 2\lambda \int_0^t \frac{dt'}{u_0} (1 - \frac{t'}{t}) \end{cases} \quad (72)$$

As in the above case of equal masses, this system differs from the system (50) by inhomogeneous terms, and this difference is $O(1)$ at large t , while the inhomogeneous term is $O(t)$. This fact indicates that $\{u_0, v_0\}$ is an asymptotic solution at $t \rightarrow \infty$.

The linearization of the system of differential equations (52), completely analogous to the linearization of equation (57), leads to linear equations, the solution of which, as well as for the case of $\mu^2 = m^2$, is expressed through cylindrical functions and has the order of $o(1)$ for large t , which confirms the conclusion that $\{u_0, v_0\}$ is the leading asymptotic solution.

In the intermediate region $m/\mu < \lambda < \mu/m$, there are no self-consistent solutions.

In fact, if either u , or v increase in absolute value, then the system of differential equations (52) dictates asymptotic behavior (53), i.e. the function v is negative at large t and, therefore, the chion propagator has a Landau-type singularity.

If the function v is positive and bounded, i.e. $0 < v \leq v_{max}$, then from the integral equation for u (see (50)), we obtain $u \geq (\mu/m - \lambda + \lambda/v_{max})t + 1$, i.e. the function u increases, and this increase again leads to the infinite increase (in the absolute value) of v in contradiction with the assumption.

A similar analysis shows that at $\lambda = \lambda_{c2} = \mu/m$ a self-consistent solution also does not exist.

Thus, in the case of unequal masses the critical point becomes a segment of critical values $\lambda_{c1} < \lambda \leq \lambda_{c2}$, in which self-consistent solutions are not available.

4 Three-particle approximation

The two-particle approximation as well as the leading term of the mean-field expansion has an incomplete crossing structure of two-particle function. For the two-particle approximation the violation is even more significant because it affects the disconnected part. However, in the same way as for the mean-field expansion, this problem is solved by considering the next approximation. The next three-particle approximation is described by a system of three equations, the first two of which are equations (9) and (11), and the third is equation (12) without Z_4 .

Just as in the two-particle approximation, Z_2 can be expressed as a functional of Δ , in the three-particle approximation Z_3 can be expressed as a functional of Z_2 and Δ :

$$Z_3 \begin{pmatrix} x & y \\ x' & y' \\ x'' & y'' \end{pmatrix} = \int dx_1 dy_1 K^{-1} \begin{pmatrix} x & y \\ x_1 & y_1 \end{pmatrix} Z_3^0 \begin{pmatrix} x_1 & y_1 \\ x' & y' \\ x'' & y'' \end{pmatrix}, \quad (73)$$

where

$$Z_3^0 \begin{pmatrix} x & y \\ x' & y' \\ x'' & y'' \end{pmatrix} \equiv - \left\{ \Delta_c(x - y'') Z_2 \begin{pmatrix} x' & y \\ x'' & y' \end{pmatrix} + \right. \\ \left. + g^2 \int dx_1 dx_2 \Delta_c(x - x_1) D_c(x_1 - x_2) Z_2 \begin{pmatrix} x_2 & x_2 \\ x' & y' \end{pmatrix} Z_2 \begin{pmatrix} x'' & y'' \\ x_1 & y \end{pmatrix} \right\} - \\ - \left\{ x' \leftrightarrow x'', y' \leftrightarrow y'' \right\}, \quad (74)$$

and K^{-1} is defined by equation (14) at $\Delta_a = \Delta_c$, $\Delta_b = \Delta$. Using this expression and the properties of the inverse kernel K^{-1} , we obtain an equation for Z_2 in the three-particle approximation, which can be written as

$$Z_2 \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} = \bar{Z}_2 \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} + \int dx_1 dy_1 \Delta_c(x - x_1) f(x_1 - y_1) \Delta(x_1 - y) \bar{Z}_2 \begin{pmatrix} y_1 & y_1 \\ x' & y' \end{pmatrix}, \quad (75)$$

wher \bar{Z}_2 is the functional of Z_2 and Δ :

$$\bar{Z}_2 \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} \equiv \Delta_c(x - y') \Delta(x' - y) + \\ + \int dx_1 dx_2 \Delta_c(x - x_1) f(x_1 - x_2) \left\{ \Delta_c(x_2 - y') Z_2 \begin{pmatrix} x_1 & y \\ x' & x_2 \end{pmatrix} + \Delta_c(x_2 - y) Z_2 \begin{pmatrix} x' & y' \\ x_1 & x_2 \end{pmatrix} + \right. \\ \left. + g^2 \int dy_1 dy_2 \Delta_c(x_2 - y_2) D_c(y_2 - y_1) \left[Z_2 \begin{pmatrix} y_1 & y_1 \\ x_1 & y \end{pmatrix} Z_2 \begin{pmatrix} x' & y' \\ y_2 & x_2 \end{pmatrix} + \right. \right. \\ \left. \left. + Z_2 \begin{pmatrix} y_1 & y_1 \\ x' & y' \end{pmatrix} Z_2 \begin{pmatrix} x_1 & y \\ y_2 & x_2 \end{pmatrix} \right] \right\} \quad (76)$$

If we approximate in $\bar{Z}_2[Z_2, \Delta]$ the dependence on Z_2 by the two-particle approximation, i.e. we consider the iterative solution, in which $\bar{Z}_2[Z_2, \Delta] \approx \bar{Z}_2[Z_2^{2PA}, \Delta]$, then we can show, using equation (9), that in this approximation the correct cross-symmetric structure is restored for the disconnected part and also for the connected part of the two-particle function.

5 Conclusions

The main result of the work is the finding of the change of asymptotic behavior in the scalar Yukawa model in the framework of two-particle approximation. The system of SDEs in the two-particle approximation has self-consistent solutions in the weak-coupling region (where a dominance of the perturbation theory is obvious) and in the strong-coupling region. The field propagators in the strong-coupling region asymptotically approach constants. It is not something unexpected, if we look at the results of studying the strong-coupling region in models of quantum

field theory. In particular, the well-known result in this direction is the conception of the ultra-local approximation (also known as the “static ultra-local approximation”), considered in the paper Caianiello and Scarpetta [22]. This exactly soluble approximation is based on removing of kinetic terms ∂^2 in the Lagrangian. As a result, all the Green functions are combinations of delta functions in the coordinate space that are constants in the momentum space. Of course, this approximation is very difficult for a physical interpretation. Nevertheless, it can be considered as a starting point for an expansion in inverse powers of the coupling constant, i.e. as a leading approximation of the strong-coupling expansion ⁵.

In this connection, it is noteworthy that the obtained solutions of the two-particle approximation in the strong-coupling region for large Euclidean momenta tend to constants. Such behavior asymptotically corresponds to the ultra-local approximation. On the other hand, our solutions are free from the interpretation problems, since for the small momenta they have the quite traditional pole behavior. This indicates that this approximation seems to be adequately described as the weak-coupling as the strong-coupling too.

In the case of equal masses, of fields a self-consistent solution of SDEs in two-particle approximation exists for any value of the coupling, including the critical value. At $g^2 = g_c^2$, the asymptotics of propagators are $1/p$, i.e. the asymptotic behavior is a medium among the free behavior $1/p^2$ at $g^2 < g_c^2$ and the constant-type behavior in strong coupling region $g^2 > g_c^2$. A sharp change of asymptotic behavior in the vicinity of the critical value is a behavior that is characteristic for a phase transition. This phase transition is not associated with a symmetry breaking, and in this sense is similar to the phase transition of “gas–liquid”. The weak-coupling region can be roughly classified as the gaseous phase and the strong-coupling region, where a kind of localization of correlators exists – to the liquid. This analogy, of course, is a quite shallow. A type and characteristics of this phase transition can be defined as the result of a detailed study using methods of the theory of critical behavior.

In the case of unequal masses, two critical values of coupling exist. In the interval between them, it is not possible to construct a solution with self-consistent ultraviolet behavior. Reasons for the existence of such an interval is currently unclear. The existence of such intermediate values of coupling can somehow reflect the metastability of the model. On the other hand, it may be an artifact of the two-particle approximation. In any case, a detailed study of the three-particle approximation of Section 4 will help to shed a light on this issue.

⁵The ultra-local approximation and the strong-coupling expansion based on it are discussed in detail in the book of Rivers [23]. For later works, see [24], [25]. For the ultra-local approximation in the bilocal source formalism, see [26].

Appendix

A linearization of nonlinear differential equations in the strong-coupling region has a sense, in general, only for large t . However, in the particular case of equation (57) at $\lambda > 1$, this linearization is available throughout the range of the t , starting from zero, i.e. a linearization of the integral equation (56) is possible. We can use this fact to estimate the values of constants A_1 and A_2 in equation (63). Performing the linearization of integral equation (56) by formula (60), we obtain for u_1 the linear integral equation

$$u_1 = -\frac{1}{\lambda-1} - a^2 \int_0^t dt' u_1(t') \left(1 - \frac{t'}{t}\right), \quad (77)$$

where a is defined by equation (62). Equation (63) tells us to deal with the Ansatz in the form

$$u_1 = \frac{A_1}{\sqrt{t}} J_1(2a\sqrt{t}), \quad (78)$$

i.e. $A_2 = 0$. The value of constant A_1 is determined by the direct substitution of the expression (78) into integral equation (77). Calculating integrals with the known formulae (see, e.g. [27]), we obtain

$$A_1 = -\frac{1}{(\lambda-1)a}. \quad (79)$$

Thus, the solution of the linearized approximation is

$$u = u_0 + u_1 = \frac{\lambda}{\lambda-1} - \frac{1}{(\lambda-1)a\sqrt{t}} J_1(2a\sqrt{t}). \quad (80)$$

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